THE POINTS OF INFLEXION OF A PLANE CUBIC CURVE.

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CONTENTS.

1. Introduction. The object of the first half of this paper is to give a self-contained and elementary exposition of the geometrical side of the theory of the inflexion points of a cubic curve without singular points. The object of the second half is to present the algebraic side of the theory, including proofs that the equation X of the ninth degree to which the problem leads is solvable by radicals, a determination of the Galois group of X for certain special cubic curves and for the general one, and a proof that X can be solved by means of a quartic and two cubic equations and hence by means of three cube roots and four square roots, no one of which can be dispensed with in general.

This interesting problem affords an excellent illustration of the complete mastery over an intricate algebraic situation which is possible by the use of Galois' theory of algebraic equations. Readers having little or no acquaintance with that theory will be able to see from this illuminating concrete example what the theory really means and what it can accomplish.

In his Traité des substitutions, Jordan laid the foundations for the applications of Galois' theory to problems of geometry and analysis, and in particular proved (p. 303) that the group of our equation X is a subgroup of the linear group L.

In his Algebra, 2d ed., vol. 2, pp. 390–418, Weber treated the problem from the geometrical and group standpoints and made (p. 416–7) some correct but unproved statements as to the actual group of equation X.

While the earlier results in the present paper are classic, the methods employed are largely novel and the exposition is especially elementary. The material in §§ 11–15 relating to the definitive determination of the Galois group is believed to be new.

2. Homogeneous Coördinates. Let x and y be the Cartesian coördi-

nates of a point referred to rectangular axes and let

$$a_i x + b_i y + c_i = 0 \qquad (i = 1, 2, 3)$$

be the equations of three straight lines L_i ($i = 1, 2, 3$) forming a triangle. Then

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

Choose the sign before the radical so that

$$p_i = \frac{a_i x + b_i y + c_i}{\pm\sqrt{a_i^2 + b_i^2}}$$

is positive for a point (x, y) inside the triangle $L_1L_2L_3$; then p_i is the length of the perpendicular from that point to L_i. The homogeneous cöordinates of a point (x, y) are three numbers x_1, x_2, x_3 such that

$$\rho x_1 = k_1 p_1, \quad \rho x_2 = k_2 p_2, \quad \rho x_3 = k_3 p_3,$$

where k_1, k_2, k_3 are constants, the same for all points, while ρ is arbitrary. Thus only the ratios of x_1, x_2, x_3 are defined. The coefficients of $k_i p_i$, which are proportional to a_i, b_i, c_i, will henceforth be denoted by the latter letters. After this slight change of notation, we have

(1) $$\rho x_i = a_i x + b_i y + c_i, \quad \Delta \neq 0 \qquad (i = 1, 2, 3).$$

Solving equations (1) by determinants, we get

$$\Delta x = \rho \Sigma A_i x_i, \quad \Delta y = \rho \Sigma B_i x_i, \quad \Delta = \rho \Sigma C_i x_i,$$

where A_i, B_i, C_i denote the cofactors of a_i, b_i, c_i in Δ. Hence

(2) $$x = \frac{A_1x_1 + A_2x_2 + A_3x_3}{C_1x_1 + C_2x_2 + C_3x_3}, \qquad y = \frac{B_1x_1 + B_2x_2 + B_3x_3}{C_1x_1 + C_2x_2 + C_3x_3}.$$

Given an equation $f(x, y) = 0$ of degree n in Cartesian coördinates, we derive by means of (2) a homogeneous equation $\phi(x_1, x_2, x_3)$ of degree n in homogeneous coördinates. In particular, any straight line is represented by an equation of the first degree in x_1, x_2, x_3, and conversely. For example, $x_1 = 0$ represents a side of the triangle of reference.

Let y_1, y_2, y_3 be the homogeneous coördinates of the same point (x, y) referred to a new triangle of reference $L_1'L_2'L_3'$. As before

(1′) $$\rho y_i = a_i' x + b_i' y + c_i' \qquad (i = 1, 2, 3),$$

where the right member equated to zero represents L_i'. Solving these as

before, we obtain x and y as linear fractional functions of y_1, y_2, y_3. Inserting these values into (1), we get formulas like

$$x_i = c_{i1}y_1 + c_{i2}y_2 + c_{i3}y_3, \quad D = | c_{ij} | \neq 0 \quad (i = 1, 2, 3). \tag{3}$$

Thus a change of triangle of reference gives rise to a linear homogeneous transformation of the homogeneous coördinates.

3. Euler's Theorem; Singular Points. Let $f(x_1, x_2, x_3)$ be a homogeneous polynomial of degree n, and let

$$t = kx_1^a x_2^b c_3^c \qquad (a + b + c = n)$$

be any term of f. Then

$$x_1 \frac{\partial t}{\partial x_1} = at, \qquad x_2 \frac{\partial t}{\partial x_2} = bt, \qquad x_3 \frac{\partial t}{\partial x_3} = ct,$$

and their sum is nt. Hence we have Euler's theorem

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} = nf. \tag{4}$$

If there is a set of solutions x_1, x_2, x_3, not all zero, of

$$\frac{\partial f}{\partial x_1} = 0, \qquad \frac{\partial f}{\partial x_2} = 0, \qquad \frac{\partial f}{\partial x_3} = 0,$$

and hence of $f = 0$, by (4), the point (x_1, x_2, x_3) is called a singular point of the curve $f = 0$. It is essential to know that the definition of a singular point is independent of the particular triangle of reference used. Under the transformation (3) of coördinates, let f become $\phi(y_1, y_2, y_3)$, so that $\phi = 0$ represents our same curve, but referred to the new triangle of reference. Then if (x_1, x_2, x_3) is a singular point of $f = 0$, the same point (y_1, y_2, y_3) is a singular point of $\phi = 0$ since

$$\frac{\partial \phi}{\partial y_j} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial y_j} = 0 \qquad (j = 1, 2, 3).$$

It is easily proved that two or more branches of the curve cross at a singular point. But we do not presuppose this geometrical interpretation, since we shall always employ the above analytic test.

4. The Hessian Covariant. The Hessian of f is

$$h = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_3} \\[2ex] \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_3} \\[2ex] \dfrac{\partial^2 f}{\partial x_3 \partial x_1} & \dfrac{\partial^2 f}{\partial x_3 \partial x_2} & \dfrac{\partial^2 f}{\partial x_3^2} \end{vmatrix}.$$

Let transformation (3), of determinant D, replace f by $\phi(y_1, y_2, y_3)$. The product hD is a determinant of the third order in which the element in the ith row and jth column is the sum of the products of the elements of the ith row of h by the corresponding elements of the jth column of D, and hence is

$$\frac{\partial^2 f}{\partial x_i \partial x_1} c_{1j} + \frac{\partial^2 f}{\partial x_i \partial x_2} c_{2j} + \frac{\partial^2 f}{\partial x_i \partial x_3} c_{3j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_j} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial y_j}\right)$$
$$= \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial y_j}\right).$$

Let D' be the determinant obtained from D by interchanging its rows and columns. By the same rule of multiplication of determinants, the element in the rth row and jth column of the determinant equal to $D' \cdot hD$ is

$$c_{1r}\frac{\partial}{\partial x_1}\left(\frac{\partial \phi}{\partial y_j}\right) + c_{2r}\frac{\partial}{\partial x_2}\left(\frac{\partial \phi}{\partial y_j}\right) + c_{3r}\frac{\partial}{\partial x_3}\left(\frac{\partial \phi}{\partial y_j}\right) = \frac{\partial}{\partial y_r}\left(\frac{\partial \phi}{\partial y_j}\right),$$

since c_{ir} is the partial derivative of x_i, given by (3), with respect to y_r. Hence

$$D^2 h = \left|\frac{\partial^2 \phi}{\partial y_r \partial y_j}\right|_{r,\, j=1,\, 2,\, 3} = \text{Hessian of } \phi.$$

Thus any linear transformation of determinant D which replaces f by ϕ also replaces D^2h by H, where h is the Hessian of f and H is the Hessian of ϕ. In other words, $H = 0$ represents the same curve as $h = 0$ when referred to the new triangle of reference. Accordingly, in investigating the relations between a curve $f = 0$ and its Hessian curve $h = 0$, we may without loss of generality assume that $f = 0$ is referred to any special triangle of reference.

5. Inflexion Points of a Cubic Curve. Let $f(x_1, x_2, x_3)$ be a homogeneous polynomial of the third degree, such that $f = 0$ has no singular point. Choose a triangle of reference having the vertex $P = (0, 0, 1)$ on the cubic curve $f = 0$. Then there is no term involving x_3^3. In the terms $rx_1x_3^2$ and $sx_2x_3^2$, r and s are not both zero, since otherwise P would be a singular point. Taking $rx_1 + sx_2$ as the new variable x_1, we see that f becomes

$$x_3^2x_1 + x_3(ax_1^2 + bx_1x_2 + cx_2^2) + \phi(x_1, x_2).$$

Replacing x_3 by $x_3 - (ax_1 + bx_2)/2$, we get

$$f_1 = x_3^2x_1 + ex_3x_2^2 + C,$$

where C is a cubic function of x_1, x_2, whose second derivative with respect

to x_i and x_j shall be designated by C_{ij}. The Hessian of f_1 is

$$h_1 = \begin{vmatrix} C_{11} & C_{12} & 2x_3 \\ C_{12} & C_{22} + 2ex_3 & 2ex_2 \\ 2x_3 & 2ex_2 & 2x_1 \end{vmatrix} = -8ex_3^3 + \cdots.$$

Thus $P = (0, 0, 1)$ is on $h_1 = 0$ if and only if $e = 0$.

Let d be the coefficient of x_2^3 in C. Then $x_1 = 0$ meets $f_1 = 0$ in the points for which $(ex_3 + dx_2)x_2^2 = 0$. Two of these points are identical with P, so that $x_1 = 0$ is tangent to $f_1 = 0$ at P. The three* points coincide if and only if $e = 0$, and P is then called an inflexion point of $f_1 = 0$ and $x_1 = 0$ the inflexion tangent at P. Hence P is on $h_1 = 0$ if and only if it is an inflexion point of $f_1 = 0$. In view of the remark at the end of § 4, we have

THEOREM 1. *Each intersection of a cubic curve $f = 0$ without a singular point with its Hessian curve $h = 0$ is an inflexion point of $f = 0$, and conversely.*

There is at least one intersection of $f = 0$, $h = 0$. For, by eliminating x_3, we get a homogeneous equation in x_1 and x_2, having therefore at least one set of solutions x_1', x_2'. Then for $x_1 = x_1'$, $x_2 = x_2'$, our equations $f = 0$, $h = 0$ have at least one common root $x_3 = x_3'$. Then (x_1', x_2', x_3') is an intersection and therefore a point of inflexion of $f = 0$.

Take this point as the vertex $(0, 0, 1)$ of a triangle of reference. As above, we get $f_1 = 0$, where now $e = 0$, $d \neq 0$. Taking a suitable multiple of x_2 as a new x_2, we have $d = 1$. We add a suitable multiple of x_1 to x_2 to delete the term $x_2^2x_1$, and get

$$F = x_3^2x_1 + C, \qquad C = x_2^3 + 3bx_2x_1^2 + ax_1^3. \tag{5}$$

Its Hessian H is the determinant h_1 for $e = 0$. Thus

$$H = 2x_1 \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} - 4x_3^2C_{22}$$

$$= 72x_1(bx_2^2 + ax_1x_2 - b^2x_1^2) - 24x_3^2x_2.$$

Eliminating x_3^2 between $F = 0$, $H = 0$, we get

$$x_2^4 + 6bx_2^2x_1^2 + 4ax_2x_1^3 - 3b^2x_1^4 = 0.$$

* If $d = e = 0$, f_1 has the factor x_1. But if $f_1 = x_1Q$,

$$\frac{\partial f_1}{\partial x_1} = Q + x_1\frac{\partial Q}{\partial x_1}, \qquad \frac{\partial f_1}{\partial x_2} = x_1\frac{\partial Q}{\partial x_2}, \qquad \frac{\partial f_1}{\partial x_3} = x_1\frac{\partial Q}{\partial x_3}$$

all vanish at a point of intersection of $x_1 = 0$, $Q = 0$, whereas $f_1 = 0$ has no singular point.

If $x_1 = 0$, then $x_2 = 0$ and we get the known inflexion point $(0, 0, 1)$. For the remaining intersections, we may set $x_1 = 1$. Then for each root of

$$r^4 + 6br^2 + 4ar - 3b^2 = 0, \tag{6}$$

we get two inflexion points $(1, r, \pm s)$, where, by $F = 0$,

$$-s^2 = r^3 + 3br + a. \tag{7}$$

In fact, no root of (6) makes $s = 0$; in other words, (6) has no double root. For, by eliminating r between the quartic and cubic equations, we get $a^2 + 4b^3 = 0$. But, the three derivatives of F with respect to x_1, x_2, x_3 are all zero at $(1, x, 0)$ if

$$x^2 + b = 0, \qquad 2bx + a = 0.$$

These are satisfied by $x = 0$ if $b = 0$, and by $x = -a/(2b)$ if $b \neq 0$, whereas $F = 0$ has no singular point. Hence

$$a^2 + 4b^3 \neq 0 \tag{8}$$

and we have proved

THEOREM 2. *Any plane cubic curve without a singular point has exactly nine distinct inflexion points.*

6. Inflexion Triangles. For a fixed root r of (6), the three points $P = (0, 0, 1)$, $(1, r, \pm s)$ are collinear, being on the line $x_2 = rx_1$. We may start with any one of the nine inflexion points in place of P. Hence they lie by threes on $9 \cdot 4/3$ lines.

THEOREM 3. *The straight line joining any two inflexion points of a cubic curve without singular points meets the curve in a new inflexion point. The nine inflexion points lie by threes upon twelve straight lines, four of which pass through any one of the nine points.*

The six inflexion points not on a particular one of these lines lie by threes upon two further lines, and the three lines are said to form an inflexion triangle. There are $12/3$ such triangles.

THEOREM 4. *The nine inflexion points lie by threes upon the sides of any one of the four inflexion triangles.*

The last theorem follows also from the important formula

$$\frac{1}{24}H + rF = (rx_1 - x_2)\left\{x_3^2 - \frac{1}{r}(rx_2 + kx_1)^2\right\}, \tag{9}$$

where $k = (r^2 + 3b)/2$. The two straight lines represented by the last factor contain the six inflexion points $(1, \rho, \pm\sigma)$, where ρ is a root $\neq r$ of (6) and, corresponding to (7),

$$-\sigma^2 = \rho^3 + 3b\rho + a.$$

The condition is $r\sigma^2 = (r\rho + k)^2$, which follows from (6) and

$$\rho^3 + r\rho^2 + (6b + r^2)\rho + r^3 + 6br + 4a = 0,$$

the equation satisfied by the roots $\neq r$ of (6).

Hence, for each root r of (6), $H + 24rF = 0$ is an inflexion triangle. A like result follows at once for our initially given cubic curve $f = 0$. For, f was transformed into F by a linear transformation of a certain determinant δ, and F is replaced by a like form by the transformation which multiplies x_3 by δ and x_1 by δ^{-2}, of determinant δ^{-1}. Hence the product of the two transformations is of determinant unity and replaces f by a form F. Hence it replaces the Hessian h of f by the Hessian H of F (§ 4). If the reduction of f to F were actually carried out, a and b would be found as certain functions of the coefficients of f. Indirect methods of finding a and b are given later.

THEOREM 5. *If $f = 0$ is any cubic curve without singular points and if h is the Hessian of f, then, for each root r of* (6), *$h + 24rf = 0$ is an inflexion triangle of $f = 0$.*

7. Usual Canonical Form of a Ternary Cubic. Consider a plane cubic curve C_3 without a singular point. Take as triangle of reference one having two vertices (1, 0, 0) and (0, 1, 0) at two inflexion points and two sides $x_2 = 0$, $x_1 = 0$ as the corresponding inflexion tangents. Let $f = 0$ be the equation of C_3 referred to this triangle. Then

$$f \equiv x_2\phi + tx_3^3 \equiv x_1\psi + tx_3^3,$$

where t is a constant, while ϕ and ψ are quadratic functions. Thus

$$f = x_1x_2l + tx_3^3, \qquad l \equiv a_1x_1 + a_2x_2 + a_3x_3. \tag{10}$$

Then $a_1a_2 \neq 0$. For, if $a_1 = 0$, the three partial derivatives of f vanish when $x_1 = 1$, $x_2 = x_3 = 0$ and (1, 0, 0) would be a singular point. Similarly, if $a_2 = 0$, (0, 1, 0) would be a singular point.

Evidently $l = 0$ is an inflexion tangent; it meets $x_3 = 0$ at an inflexion point distinct from (1, 0, 0) and (0, 1, 0), since $a_1a_2 \neq 0$. We thus have another proof of the first part of Theorem 3.

Take as new variables z_1, z_2, x_3, where

$$a_1x_1 = \omega z_1 + \omega^2 z_2 + c, \qquad a_2x_2 = \omega^2 z_1 + \omega z_2 + c, \qquad c = -\tfrac{1}{3}a_3x_3,$$

where ω is an imaginary cube root of unity. Then

$$-l = z_1 + z_2 + c,$$

$$-a_1a_2x_1x_2l = \begin{vmatrix} c & z_1 & z_2 \\ z_2 & c & z_1 \\ z_1 & z_2 & c \end{vmatrix} = z_1^3 + z_2^3 + c^3 - 3z_1z_2c,$$

$$-a_1a_2f = z_1^3 + z_2^3 - (a_1a_2t + \tfrac{1}{27}a_3^3)x_3^3 + a_3z_1z_2x_3.$$

The coefficient of x_3^3 is not zero, since otherwise the three partial derivatives of f would vanish for $z_1 = z_2 = 0$ and there would be a singular point. Hence by taking a suitable multiple of x_3 as z_3, we get

$$f = \alpha(z_1^3 + z_2^3 + z_3^3) + 6\beta z_1 z_2 z_3. \tag{11}$$

We saw that $x_3 = 0$ intersects the cubic (10) in three inflexion points. Hence $z_3 = 0$ intersects (11) in three inflexion points. By symmetry, the same is true of $z_1 = 0$ and of $z_2 = 0$. Since a non-vanishing coördinate may be taken to be unity, we obtain at once the nine points of inflexion of (11):

$$\begin{array}{ccc} (\ \ 0,\ \ 1,\ -1) & (\ \ 0,\ \ 1,\ -\omega) & (\ \ 0,\ \ 1,\ -\omega^2) \\ (-1,\ \ 0,\ \ 1) & (-\omega,\ \ 0,\ \ 1) & (-\omega^2,\ \ 0,\ \ 1) \\ (\ \ 1,\ -1,\ \ 0) & (\ \ 1,\ -\omega,\ \ 0) & (\ \ 1,\ -\omega^2,\ \ 0). \end{array} \tag{12}$$

The Hessian of (11) is $6^3 g$, where

$$g = -\alpha\beta^2(z_1^3 + z_2^3 + z_3^3) + (\alpha^3 + 2\beta^3)z_1 z_2 z_3. \tag{13}$$

The functions f and g are linearly independent since

$$d \equiv \alpha(\alpha^3 + 8\beta^3) \neq 0. \tag{14}$$

In fact, the partial derivatives of f vanish if

$$\alpha z_1^2 + 2\beta z_2 z_3 = \alpha z_2^2 + 2\beta z_1 z_3 = \alpha z_3^2 + 2\beta z_1 z_2 = 0,$$

which are satisfied when $z_1 = z_2 = \beta$, $z_3 = -\alpha/2$, if $\alpha^3 + 8\beta^3 = 0$, and when $z_1 = z_2 = 1$, $z_3 = 0$ if $\alpha = 0$. In view of (14), the intersections of $f = 0$, $g = 0$ are those of $f = 0$, $z_1 z_2 z_3 = 0$. Hence the nine points (12) are the only inflexion points of $f = 0$.

The three points in the ith row of table (12) lie on the line $z_i = 0$ in view of their origin. The three in the first, second and third columns lie in the respective lines

$$z_1 + z_2 + z_3 = 0, \qquad \omega^2 z_1 + \omega z_2 + z_3, \qquad \omega z_1 + \omega^2 z_2 + z_3 = 0. \tag{15}$$

If we multiply the first coördinate of each of the points in the respective columns (12) by ω, we obtain three sets of three points, the first set being those in the main diagonal of table (12) and the other two sets being those in the positions of the elements of a determinant of the third order which yield the remaining two positive terms. Hence they lie by threes upon the lines derived from (15) by replacing z_1 by $\omega^2 z_1$.

But if we multiply the first coördinate of each point in the respective columns by ω^2, we get those corresponding to negative terms of the determinant, which therefore lie by threes upon the lines derived from (15) by replacing z_1 by ωz_1.

We thus have a new proof of the second part of Theorem 3 as well as of Theorem 4. Moreover, we now see just how the nine inflexion points are distributed into twelve sets of three collinear points. This distribution can be described most briefly by employing for the nine inflexion points the notation $[\xi\eta]$, where ξ and η take the values 0, 1, 2 independently. To the table (12) we make correspond

$$\begin{matrix} [00] & [12] & [21] \\ [11] & [20] & [02] \\ [22] & [01] & [10]. \end{matrix} \tag{16}$$

THEOREM 6. *Three inflexion points* $[\xi_1, \eta_1]$, $[\xi_2, \eta_2]$, $[\xi_3, \eta_3]$ *are collinear if and only if*

$$\xi_1 + \xi_2 + \xi_3 \equiv 0, \qquad \eta_1 + \eta_2 + \eta_3 \equiv 0 \qquad (mod\ 3). \tag{17}$$

From (11) and (13), we get

$$dz_1z_2z_3 = \alpha g + \alpha\beta^2 f, \qquad d\Sigma z_1^3 = -6\beta g + (\alpha^3 + 2\beta^3)f. \tag{18}$$

By the first, $g + \beta^2 f = 0$ gives one inflexion triangle. The product of the left members of the sides (15) of another inflexion triangle is

$$\begin{vmatrix} z_1 & z_2 & z_3 \\ z_3 & z_1 & z_2 \\ z_2 & z_3 & z_1 \end{vmatrix} = \Sigma z_1^3 - 3z_1z_2z_3 = \frac{(\alpha + 2\beta)}{d}[-3g + (\alpha - \beta)^2 f],$$

by (18). The remaining two inflexion triangles are obtained from the last by replacing z_1 by $\omega^2 z_1$ or ωz_1. In view of (18), we have only to replace α by $\omega^2\alpha$ or $\omega\alpha$ in our final result. We thus get the

LEMMA. *The four inflexion triangles of* (11) *are* $3g + rf = 0$, *where*

$$r = 3\beta^2, \qquad -(\alpha - \beta)^2, \qquad -(\omega\alpha - \beta)^2, \qquad -(\omega^2\alpha - \beta)^2. \tag{19}$$

The quartic having these four numbers as roots is easily found. First, α, $\omega\alpha$, $\omega^2\alpha$ are the roots of $x^3 - \alpha^3 = 0$. Diminishing each root by β, we obtain the equation

$$(y + \beta)^3 - \alpha^3 = 0.$$

Then $(\alpha - \beta)^2$, etc., are the roots of a cubic obtained by transposing the terms of even degree, squaring, and replacing y^2 by z. By inspection, we get

$$z(z + 3\beta^2)^2 = (-3\beta z + \alpha^3 - \beta^3)^2.$$

Hence the last three numbers (19) are the roots of the cubic obtained by replacing $-z$ by r. Multiply the new cubic by $r - 3\beta^2$. We get the

desired quartic (6), in which

$$(20) \qquad b = \beta(\alpha^3 - \beta^3), \qquad a = \tfrac{1}{4}\alpha^6 - 5\alpha^3\beta^3 - 2\beta^6.$$

We have therefore a second proof of Theorem 5.

8. Nature of the Coefficients of Quartic (6). For two special cubic curves (5) and (11), we have determined the coefficients a and b of (6). It is important for the sequel to know that, just as in these two cases, a and b are always rational functions of the coefficients of f.

First, the four roots of (6) are the only values of r for which $\phi \equiv h + 24rf$ has a linear factor l. For, $l = 0$ meets $f = 0$ in three points on $h = 0$, which are therefore inflexion points. Thus l is one of the twelve lines through three inflexion points, and there are two companion lines l_1 and l_2 such that $ll_1l_2 = h + 24tf$, where t is a root of (6). By hypothesis, $lQ = \phi$. By subtraction, we see that, if $r \neq t$, f has the factors l and $Q - l_1l_2$ and hence has a singular point (foot-note in § 5).

Next, if $x_1 - dx_2 - ex_3$ is a linear factor of ϕ and we replace x_1 by $dx_2 + ex_3$ in ϕ, we obtain a cubic function of x_2, x_3, whose four coefficients must vanish. Eliminating d and e, we obtain two conditions in which r and the coefficients of f enter rationally and integrally. If we free the greatest common divisor of their left members from multiple factors, we must obtain a quartic function of r whose coefficients are rational in those of f.

THEOREM 7. *The four inflexion triangles of a cubic curve $f = 0$ without singular points depend upon a quartic* (6) *whose coefficients a and b are rational* functions of the coefficients of f.*

9. The Equation X for the Abscissas of the Nine Inflexion Points. Let R denote the set or domain of all rational functions with rational coefficients of the coefficients of the equation $f = 0$ of the cubic curve. After applying a linear transformation on x_1, x_2, x_3, with coefficients in R, we may assume that $x_3 \neq 0$ for each inflexion point (we have only to take a new triangle of reference whose side $x_3 = 0$ does not pass through a point of inflexion). Pass from homogeneous to Cartesian coördinates by setting $x = x_1/x_3$, $y = x_2/x_3$. After rotating the axes through a suitable angle with a rational sine and cosine, we may assume that the y-axis is not parallel to any line joining two inflexion points. Then the abscissas $x_1, \cdots, x_9$ of the inflexion points are distinct, and the equations (with coefficients in R) of the curve and its Hessian curve uniquely determine the ordinate y_i of the inflexion point (x_i, y_i) as a rational function of x_i with coefficients in R. Thus, by eliminating y^3 and y^2 between the equations

* In fact, integral functions, since otherwise a root r would be infinite for certain f's. The expressions for $S = b$, $T = 4a$ are invariants of f, first computed by Aronhold and given in full in Salmon's Higher Plane Curves, §§ 221–2. No use is made of these long invariants in this paper.

of these two curves, we obtain y expressed as a rational function of x in R with denominator not zero at an inflexion point. By substituting this expression for y into $f = 0$, we obtain the equation X for the nine abscissas of the inflexion points. The relation

$$\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} = 0$$

between the coördinates of three collinear inflexion points gives a rational relation $\psi(x_i, x_j, x_k) = 0$, with coefficients in R, between their abscissas. By means of $\psi = 0$ and X, we can express x_i as a rational function of x_j and x_k with coefficients in R (Theorem 3). Consider, conversely, three inflexion points whose abscissas satisfy $\psi = 0$ and let the line joining two of them (x_j, y_j) and (x_k, y_k) meet the curve at (x, y); the latter is an inflexion point, so that $x = x_i$ and hence $y = y_i$.

THEOREM 8. *Three inflexion points are collinear if and only if their abscissas satisfy the rational relation $\psi = 0$ with coefficients in R.*

Let G be the group of permutations of $x_1, \cdots, x_9$ such that every rational function of $x_1, \cdots, x_9$ with coefficients in R which is unaltered by all of the permutations of G equals a number in the domain R, and conversely. Then G is called the Galois group of equation X for the domain R.

Let a permutation of G replace three roots x_1, x_2, x_3 for which $\psi(x_1, x_2, x_3) = 0$ by the roots x_r, x_s, x_t. By the converse property just stated, $\psi(x_r, x_s, x_t) = 0$. Hence Theorem 8 yields

THEOREM 9. *Every permutation of the Galois group G of the equation X for the nine abscissas of the inflexion points replaces the abscissas of any three collinear inflexion points by the abscissas of three collinear inflexion points.*

10. Galois Group G a Linear Group. Henceforth, we shall denote the abscissas by the nine symbols $[\xi\eta]$ used in (16) for the corresponding inflexion points. By Theorem 9, G is a subgroup of the group L of those permutations on the nine roots $[\xi\eta]$ which replace three roots forming a row, column, positive or negative term of the determinant of table (16) by a set of three such roots, or, by Theorem 6, which replace three roots satisfying congruences (17) by three roots also satisfying them.

For each set of integers $a, \cdots, C$, the linear substitution

$$(21) \quad \xi' \equiv a\xi + b\eta + c, \qquad \eta' \equiv A\xi + B\eta + C, \qquad \begin{vmatrix} a & b \\ A & B \end{vmatrix} \not\equiv 0 \pmod 3$$

gives rise to a permutation of the nine roots $[\xi\eta]$. For, it replaces two distinct roots $[\xi_1\eta_1]$ and $[\xi_2\eta_2]$ by distinct roots $[\xi_1'\eta_1']$ and $[\xi_2'\eta_2']$. If the

latter were identical, then

$$a(\xi_1 - \xi_2) + b(\eta_1 - \eta_2) \equiv 0, \quad A(\xi_1 - \xi_2) + B(\eta_1 - \eta_2) \equiv 0 \pmod 3,$$

and, in view of the determinant of (21), $\xi_1 - \xi_2 \equiv \eta_1 - \eta_2 \equiv 0$, contrary to hypothesis. Moreover, the resulting permutation belongs to the group L. For, if $[\xi_i\eta_i]$ where $i = 1, 2, 3$, are distinct roots satisfying congruences (17), then $[\xi_i'\eta_i']$ satisfy (17), since

$$\sum_{i=1}^{3} \xi_i' = a\sum_{i=1}^{3} \xi_i + b\sum_{i=1}^{3} \eta_i + 3c \equiv 0, \qquad \sum_{i=1}^{3} \eta_i' \equiv 0 \pmod 3.$$

Conversely, every permutation P of L is induced by a linear substitution (21) on the indices. For, if P replaces [00] by $[cC]$ then $P = P_1S$, where P_1 is a permutation of L which leaves [00] unaltered and S is that induced by

(22) $$\xi' \equiv \xi + c, \qquad \eta' \equiv \eta + C \pmod 3.$$

Let P_1 replace [10] by $[aA]$, so that a and A are not both zero. Then we can find two integers b and B such that $aB - bA$ is not divisible by 3 (if $a = 1$ or 2, take $b = 0$, $B = 1$; if $A = 1$ or 2, take $b = 1$, $B = 0$). Hence $P_1 = P_2S_1$, where S_1 is the permutation induced by

(23) $$\xi' \equiv a\xi + b\eta, \qquad \eta' \equiv A\xi + B\eta, \qquad aB - bA \not\equiv 0 \pmod 3,$$

while P_2 is the permutation of L which alters neither [00] nor [10] and hence not [20] in view of the diagonal term of table (16). Let P_2 replace [01] by $[de]$, so that $e \neq 0$. Then $P_2 = P'S_2$, where S_2 is the permutation induced by

$$\xi' \equiv \xi + d\eta, \qquad \eta' \equiv e\eta,$$

while P' is a permutation of L which leaves fixed [00], [10], [20] and [01], and hence [22] by the third row of (16), then [11] and [12] by the first and second columns, and finally [21] and [02] by the first and second rows. Thus P' is the identity and $P = P_1S = P_2S_1S = S_2S_1S$. Since the product of three linear substitutions (21) is such a substitution, P is induced by a linear substitution.

Now $[cC]$ was any one of 3×3 roots, $[aA]$ any one of $3^2 - 1$ roots, $[de]$ any one of 3×2 roots. We thus have

THEOREM 10. *The Galois group G is a subgroup of the group L of the* $9 \cdot 8 \cdot 6$ *linear substitutions* (21) *on the indices* ξ, η.

11. Structure of the Linear Group L. There are exactly $3^2 - 1$ incongruent linear homogeneous functions of ξ and η with integral coefficients modulo 3, viz.,

$$\pm\,\xi, \qquad \pm\,\eta, \qquad \pm\,(\xi + \eta), \qquad \pm\,(\xi - \eta).$$

Hence they are permuted by the linear homogeneous substitutions (23), which form a group H. The same is true of their squares $\xi^2, \cdots$. The identity I and

$$J: \qquad \xi' \equiv -\xi, \qquad \eta' \equiv -\eta$$

are the only substitutions (23) which leave unaltered each of the four squares. For, if the sign of ξ is changed, while η is not changed, or vice versa, then $(\xi + \eta)^2$ becomes $(\xi - \eta)^2$. Hence there is a (2, 1) correspondence between the 48 substitutions of H and the permutations on the four squares, so that H corresponds to the symmetric group G_{24} on four letters. As well known, G_{24} has an invariant (self-conjugate) subgroup G_{12}, which has an invariant subgroup G_4, with in turn an invariant G_2. It follows that H has a series of subgroups of orders 24, 8, 4, 2, each invariant in the preceding, so that H is a solvable group.

The group T of the nine "translations" (22) is invariant in L. In fact, (23) transforms (22) into the translation

$$\xi' \equiv \xi + ac + bC, \qquad \eta' \equiv \eta + Ac + BC \qquad (\text{mod } 3).$$

Since T has an invariant subgroup of order 3, we have

THEOREM 11. *The linear group L is a solvable group.*

12. Equation X Solvable by Radicals. If we make use of the well known theorems that any subgroup of a solvable group is a solvable group and that an equation is solvable by radicals if its group is a solvable group, we obtain, from Theorems 10 and 11,

THEOREM 12. *If R is the domain defined by the coefficients of the equation $f = 0$ of a cubic curve without singular points, and if X is the equation for the abscissas of its nine inflexion points, the Galois group G of X for R is a solvable group, and equation X is solvable by radicals.*

But we need not presuppose the theorem that any subgroup of a solvable group is solvable in order to conclude that X is solvable by radicals. For, we shall prove that G is identical with the solvable group L in case the coefficients of f are independent variables. Since therefore X is then solvable by radicals, it will continue to be solvable by radicals when one or more of the variable coefficients of f have been given special values. However, the use of the theorem that the subgroups of a solvable group are solvable enables us to derive the important Theorem 12 without entering upon the difficult question of the actual determination of the Galois group G. Similarly,* we may draw important conclusions as to the number of real bitangents to a quartic curve or real lines on a cubic surface, if we have merely proved that the group G of the corresponding equation is a subgroup of a certain group and have not actually determined G.

* Dickson, *Annals of Mathematics*, vol. 6 (1905), p. 141.

13. Group of the Resolvent Quartic (6). If $r_1, \cdots, r_4$ are the roots of (6), it is shown in texts on the theory of equations* that

$$y_1 = r_1r_2 + r_3r_4, \qquad y_2 = r_1r_3 + r_2r_4, \qquad y_3 = r_1r_4 + r_2r_3$$

are the roots of

$$y^3 - 6by^2 + 12b^2y - 16a^2 - 72b^3 = 0.$$

Setting $y = z + 2b$, we obtain the reduced cubic

$$z^3 = D \equiv 16(a^2 + 4b^3),$$

where $D \neq 0$ by (8). The discriminant of (6) is known to equal the discriminant of $z^3 - D$ and hence is $-27D^2$.

Consider the case $a = 1$, $b = -1$. Then $D = -48$ is not the cube of a rational number, so that each y_i is irrational. Hence, in view of Ferrari's method of solving quartic equations, our present form of (6)

$$r^4 - 6r^2 + 4r - 3 = 0 \tag{6'}$$

is not the product of two quadratic factors with rational coefficients. Nor has it a rational root, since no root equals ± 1, ± 3. Hence (6′) is irreducible in the domain $R(1)$ of rational numbers. The group of (6′) for $R(1)$ is therefore transitive; it is not a subgroup of the G_8 which leaves y_i unaltered; nor is it the alternating group G_{12} since the discriminant of (6′) was seen to be $-27D^2$ and hence not the square of a rational number. Hence the group of (6′) for $R(1)$ is the symmetric group G_{24}. Since a 24 valued function of the roots of (6′) is therefore a root of an irreducible equation of degree 24, the same is true of the quartic (6) when the coefficients of f are arbitrary variables.

THEOREM 13. *Let the coefficients of the equation $f = 0$ of the cubic curve be independent variables and let R be the domain defined by them. Then the group of* (6) *for R is the symmetric group G_{24}.*

COROLLARY. *The adjunction of the four roots of* (6) *to R reduces the group G of equation X for R to an invariant subgroup Σ of index* 24 *under G.*

The Corollary follows from Theorem 13 and the Theorem of Jordan,† since the adjunction of all nine roots of X reduces the group of (6) to the identity. For, the abscissas and hence the ordinates (§ 9) of the inflexion points are in the enlarged domain. Thus the ratios of the coefficients of the equation of the line through collinear inflexion points are in that domain. Since three such lines form an inflexion triangle, it follows from Theorem 5 that each root r of (6) is in the enlarged domain.

* Cf. Dickson's Elementary Theory of Equations, 1914, p. 39.

† Traité des substitutions, p. 269; cf. Dickson's Introduction to the Theory of Algebraic Equations, p. 81.

14. Group G for a General Cubic Curve. Adjoin the root of (6) associated with the inflexion triangle whose sides are determined by the sets of inflexion points

$$[00],\ [01],\ [02];\ [10],\ [11],\ [12];\ [20],\ [21],\ [22], \tag{24}$$

corresponding to the negative terms of the expansion of the determinant of (16). Then G reduces to the subgroup which permutes these three sets (24). This is true of the substitution which merely changes the sign of ξ, or that which merely changes the sign of η, or of

$$\xi' \equiv \xi + c, \qquad \eta' \equiv A\xi + \eta + C \qquad (\text{mod } 3),$$

which replaces the kth set (24) by the $(k + c)$th set (24). These three types evidently generate the group of the $4 \cdot 3^3$ substitutions (21) with $b \equiv 0$, $aB \not\equiv 0$, A, c, C arbitrary modulo 3, whose index under the total group L is 4. Since there are four inflexion triangles, we now have all the substitutions of L which permute the sets (24).

We now adjoin also the root of (6) which is associated with the inflexion triangle determined by the three sets of inflexion points

$$[00],\ [10],\ [20];\ [01],\ [11],\ [21];\ [02],\ [12],\ [22],$$

corresponding to the positive terms of the expansion of the determinant of (16). Since these sets are derived from the sets (24) by the interchange of ξ and η, the largest subgroup of L which permutes these sets is the group of substitutions (21) with $A \equiv 0$. After both adjunctions, the group is contained in that composed of

$$\xi' \equiv a\xi + c, \qquad \eta' \equiv B\eta + C, \qquad aB \not\equiv 0 \quad (\text{mod } 3). \tag{25}$$

Of these, the substitutions

$$\xi' \equiv a\xi + c, \qquad \eta' \equiv a\eta + C \qquad (a = 1 \text{ or } -1) \tag{26}$$

permute the rows (16) and also the columns, while $\xi' \equiv \xi$, $\eta' \equiv -\eta$ interchanges the rows and columns. The latter is therefore true of (25) if $a \equiv -B$. We have now proved

THEOREM 14. *After the adjunction of all four roots of* (6), *the group G of X reduces to a subgroup Σ of the group of the* 18 *substitutions* (26).

Information concerning Σ is gained from special cubic curves.

First, we take $b = 0$, $a = -2$ in (5). Then (6) becomes $r(r^3 - 8) = 0$. Hence after the adjunction of the four roots, the domain is $R(\omega)$, where ω is an imaginary cube root of unity. But the sides of any one of the inflexion triangles involve the radical $\sqrt{2}$, which is not in $R(\omega) = R(\sqrt{-3})$. For $r = 0$, the triangle is given by the Hessian $-24x_2(x_3^2 + 6x_1^2)$ of F. For

$r \neq 0$, the triangle is given by (9), two of whose sides involve the radical $\sqrt{r}$ and hence the new radical $\sqrt{2}$. Thus Σ is here of order 2, while the group G of X for the domain of the rational numbers is of order 4.

Next, we employ the special cubic curve

$$f = x_1^3 + 2x_2^3 + 4x_3^3 + 6x_1x_2x_3. \tag{27}$$

It is reduced to the canonical form (11) with $\alpha = 2$, $\beta = 1$, by

$$x_1 = \sqrt[3]{2}z_1, \qquad x_2 = z_2, \qquad x_3 = z_3/\sqrt[3]{2},$$

a transformation of determinant unity. The roots (19) of (6) are in the domain $R(\omega)$. By (15) and the remarks following it, the sides of the four inflexion triangles involve the single new irrationality $\sqrt[3]{2}$. Hence Σ is here of order 3, while G is of order 6.

Consider the cubic curve the coefficients of whose equation are independent variables and let R be the domain defined by them. In view of the last results, the order of the group Σ is divisible by 2 and 3. Since (26) is of period 2 if $a = -1$, a substitution (26) of period 3 has $a = +1$. Interchanging the variables if necessary, we may assume that ξ is altered. Then the substitution or its square is $\xi' \equiv \xi + 1$, $\eta' \equiv \eta + t$. Introducing ξ and $\eta - t\xi$ as new variables, we obtain a group Σ_1 conjugate with Σ under L and having the substitution $\xi' \equiv \xi + 1$, $\eta' \equiv \eta$. The only substitutions (21) commutative with this one are those with $a \equiv 1$, $A \equiv 0$, $B \not\equiv 0$, and b, c, C arbitrary, $2 \cdot 3^3$ in number. It is transformed into its inverse by $\xi' = -\xi$, $\eta' = \eta$. Hence exactly $4 \cdot 3^3$ substitutions of L transform into itself the cyclic group of order 3 generated by our substitution. If Σ were of order 6, it would contain a single cyclic group of order 3, which is invariant under G, since Σ is. By the corollary in § 13, G would then be of order 24×6, which exceeds $4 \cdot 3^3$. From this contradiction with the preceding result, we conclude that Σ contains all 18 substitutions (26). The order of G thus equals that of H.

THEOREM 15. *If the coefficients of a cubic curve $f = 0$ are independent variables, the group of the equation X for the abscissas of the nine inflexion points $[\xi\eta]$, $\xi, \eta = 0, 1, 2$, for the domain of the coefficients of f, is the group L of all linear substitutions on ξ and η.*

15. Determination of the Inflexion Points. Let the coefficients of the equation of the cubic curve be independent variables. The solution of quartic (6) requires the extraction of a cube root and three square roots, since its group is G_{24}. After the adjunction of its four roots, the group of X is that of the 18 substitutions (26). The product $h + 24rf$ of the linear functions which vanish at the sides of an inflexion triangle (Theorem 5) has as coefficients numbers in the enlarged domain. The determination of the

linear factors requires the solution of a cubic equation. For example, for (27) and $r = -1$, the factors are $x_1 + \lambda x_2 + 2\lambda^{-1}x_3$, where $\lambda^3 = 2$. Consider the inflexion triangle determined by (24); after the adjunction of the roots of the corresponding cubic equation, the group permutes the roots in each set (24). The only substitutions (26) having this property are $\xi' \equiv \xi$, $\eta' \equiv \eta + C$, which form a group G_3. The group of this resolvent cubic is therefore of order 6. In the new domain, the group of the corresponding resolvent cubic for another inflexion triangle is G_3, whose solution requires only the extraction of a cube root. After the adjunction of the latter, we have the sides of two inflexion triangles, and their intersections give the nine inflexion points.

THEOREM 16. *The inflexion points can be found by solving a quartic and two cubic equations, and the solution involves three cube roots and four square roots. For a general cubic curve, no one of these radicals can be avoided or expressed in terms of the others.*

THE UNIVERSITY OF CHICAGO,
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